The Problem of Runaway Solutions in the Lorentz-Dirac Theory

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A rigorous non-existence proof for runaway solutions in the finite-size model of the electron is given. Since a consistent point limit, such as the Lorentz-Dirac equation claims to be, should not exhibit features which are completely missing in the more general theory of finite extension, it is proposed that the point-like approximation of the finite-size theory be the integrodifferential formulation of the Lorentz-Dirac theory. This point of view is supported by a new discussion of the hyperbolic motion in the latter theory.

I. Introduction

As is well-known, one of the main objections to the Lorentz-Dirac theory of the radiating electron aims at the existence of runaway solutions being admitted by the basic equation of motion of this theory: without presence of external forces the electron is able to speed up towards the velocity of light; a phenomenon, which never has been observed in reality.

Rohrlich has extensively represented this problem in his book [1] and he proposes to exclude a priori these unphysical solutions by the postulate of well-defined initial and final states. But, of course, one wants to have a theory, in which runaway solutions do not exist and in which therefore there would be no need to discuss them away with a lot of persuasive power.

What sort of theory could remove the undesired effect? It has ever been speculated that the runaway solutions are due to the point limit, which requires a mass renormalization in one way or the other. The following work claims to prove this speculation to be correct, at least as far as the finite-size theory of the radiating electron recently developped [2, 3] is concerned. It can be shown explicitly in this model that ruanaway solutions do not exist as long as one retains the finite-size of the particle.

Moreover, it is exhibited by means of hyperbolic motion that the Lorentz-Dirac theory in its integro-differential form (but not in its differential formulation) provides a reasonable point-like approximation for the finite-size theory.

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II. Short Review of the Finite-size Theory and the Problem of Runaway Solutions

According to the theory of the finite-size electron [2] the equation of notion reads in the force-free case

$$\dot{P}_{\mathrm{b}}{}^{\mu} + \dot{P}_{\mathrm{r}}{}^{\mu} = 0\,, \qquad \text{(II, 1)}$$

whereby the following expressions for the bound ($P_{\mathrm{b}}{}^{\mu}$) and the emitted ($P_{\mathrm{r}}{}^{\mu}$) four-momenta have been used

$$P_{\mathrm{b}}{}^{\mu} = \frac{1}{c}\,\frac{Z^2}{2\,\varrho}\left\{\frac{4}{3}\,(\hat{u}\,u)\,\hat{u}^{\mu} - \frac{1}{3}\,u^{\mu}\right\}, \qquad \text{(II, 2a)}$$

$$\dot{P}_{\mathrm{r}}{}^{\mu} = -\frac{1}{c}\,\frac{2}{3}\,Z^2 (\hat{\dot{u}}\,\hat{\dot{u}})\,\hat{u}^{\mu}\,. \qquad \text{(II, 2b)}$$

Differentiation with respect to the proper time s, which has the dimension of a length, is indicated by a dot and quantities like $\hat{u}^{\lambda}$ or $\hat{\dot{u}}^{\lambda}$ are shifted backwards in (proper) time by the constant amount Δs

$$\hat{u}^{\lambda}{}_{(s)} := u^{\lambda}{}_{(s-\Delta s)}\,; \quad \hat{\dot{u}}^{\lambda}{}_{(s)} := \dot{u}^{\lambda}{}_{(s-\Delta s)} \quad \text{etc.}$$

The numerical value of the proper time interval Δs has been put equal to the classical electron radius r_0, which is to be determined from

$$Z^2/2\,r_0 = m_0\,c^2$$

with m_0 being the experimental electron rest mass. The covariant generalization of the classical radius is assumed in this model to be the acceleration-dependent electron radius ϱ [emerging in (II, 2a)], defined by ($u^{\lambda} \equiv \mathrm{d}z^{\lambda}/\mathrm{d}s$; $u^{\lambda} u_{\lambda} = 1$):

$$\varrho = (z - \hat{z}) \cdot u\,. \qquad \text{(II, 2c)}$$

If the electron world line $x^{\lambda} = z^{\lambda}{}_{(s)}$ is curved only weakly (which is not the case for runaway solutions), those quantities in (II, 2) like $\hat{u}^{\lambda}$, $\hat{\dot{u}}^{\lambda}$ etc., which are shifted backwards in time, may be expanded in a Taylor series about the reference point $z^{\lambda}{}_{(s)}$

$$\hat{u}^{\lambda} = u^{\lambda} - \Delta s\, \dot{u}^{\lambda} + \tfrac{1}{2}(\Delta s)^2\, \ddot{u}^{\lambda} \ldots \quad \text{(II, 3a)}$$

$$\hat{\dot{u}}^{\lambda} = \dot{u}^{\lambda} - \Delta s\, \ddot{u}^{\lambda} + \tfrac{1}{2}(\Delta s)^2\, \dddot{u}^{\lambda} \ldots \quad \text{(II, 3b)}$$

and after insertion into the equation of motion (II, 1) the terms of higher power in Δs may be neglected, so that the famous Lorentz-Dirac equation[4] arises

$$\frac{Z^2}{2\,\Delta s}\, \dot{u}^{\lambda} - \frac{2}{3}\, Z^2 \{\ddot{u}^{\lambda} + (\dot{u}\,\dot{u})\, u^{\lambda}\} = 0\,, \quad \text{(II, 4)}$$

containing only terms up to Δs^0 inclusively. Usually, a mass renormalization is performed in the first term on the left of (II, 4), in order to have Δs vanishing exactly, whereby the neglected terms of higher power in Δs would vanish exactly, too; and then one could argue, that (II, 4) should be the exact equation of motion for a point-like particle. A philosophy of this sort was originally used by Dirac[4] for the derivation of the Eq. (II, 4) with $Z^2/2\,\Delta s$ substituted by $m_0\, c^2$ as a consequence of mass renormalization *.

But Peierls[5] has pointed out recently, that in the case of runaway solutions the neglected terms, which contain higher powers in Δs and higher derivatives in the four-velocity $\{u^{\lambda}\}$, may not be ignored, because they all are of the same order of magnitude. Therefore it seems reasonable to speculate that those unphysical effects like pre-acceleration and runaway solutions are unavoidably connected with the point limit but are not present in a finite-size theory, because such a theory does account for the higher order terms by virtue of the nonlocality of the equation of motion. In this respect, the paper of Daboul[8] deserves attention, who has shown in the nonrelativistic case, that a finite number of higher order terms, added to the pointlike equation of motion, does not prohibit the existence of runaway solutions. It seems that only a nonlocal theory, being equivalent to an infinite number of higher derivatives, is able to exclude the unphysical solutions.

In the following, we shall prove that this supposition is true, at least in the case of our finite-size model of the radiating electron[2]. We restrict ourselves to onedimensional (linear) motion and we shall show that the extended electron can indeed only move with constant four-velocity, if there are no external forces acting.

* In Dirac's work[4] as well as in Rohrlich's book[1], which is also referred to by Teitelboim[6], the derivation of the equation of motion (II, 4) was managed by use of tedious series expansions being absolutely superfluous in our finite-size model[1] as a consequence of the "retarded geometry". This important simplification was meanwhile recognized by Tabensky and Villarroel[7], too.

III. Basic Equations for the Proof

For the proof it is more convenient to start from two scalar equations instead from the four-vector equation (II, 1). In order to obtain the first one of these two scalar equations, we multiply the force-free equation of motion (II, 1) with $\{P_{b\mu}\}$:

$$0 = (\dot{P}_{b}{}^{\mu}\, P_{b\mu}) + \dot{P}_{r}{}^{\mu}\, P_{b\mu}) = \frac{d}{ds}\left[\frac{1}{2}\,(P_{b}{}^{\mu}\, P_{b\mu})\right] + (\dot{P}_{r}{}^{\mu}\, P_{b\mu})\,. \quad \text{(III, 1)}$$

Here one finds easily with (II, 2a, b)

$$(P_{b}{}^{\mu}\, P_{b\mu}) = \frac{1}{c^2}\, \frac{Z^4}{36\,\varrho^2}\, \{8\,(u\,\hat{u})^2 + 1\} \quad \text{(III, 2)}$$

and

$$(\dot{P}_{r}{}^{\mu}\, P_{b\mu}) = -\frac{1}{c^2}\, \frac{Z^4}{3\,\varrho}\, (\hat{\dot{u}}\,\hat{\dot{u}})\,(u\,\hat{u})\,. \quad \text{(III, 3)}$$

With these results, Eq. (III, 1) reads as follows

$$\frac{d}{ds}\left[\frac{8\,(u\,\hat{u})^2 + 1}{\varrho^2}\right] = 24\,(\hat{\dot{u}}\,\hat{\dot{u}})\, \frac{(u\,\hat{u})}{\varrho}\,. \quad \text{(III, 4)}$$

The second scalar equation is obtained by multiplying (II, 1) with $\{\hat{u}_{\mu}\}$

$$0 = (\dot{P}_{b}{}^{\mu}\, \hat{u}_{\mu}) + (\dot{P}_{r}{}^{\mu}\, \hat{u}_{\mu}) = \frac{d}{ds}\,[P_{b}{}^{\mu}\, \hat{u}_{\mu}] - (P_{b}{}^{\mu}\, \hat{\dot{u}}_{\mu}) + (\dot{P}_{r}{}^{\mu}\, \hat{u}_{\mu}) \quad \text{(III, 5)}$$

and with $\{u_{\mu}\}$

$$0 = (\dot{P}_{b}{}^{\mu}\, u_{\mu}) + (\dot{P}_{r}{}^{\mu}\, u_{\mu}) = \frac{d}{ds}\,[P_{b}{}^{\mu}\, u_{\mu}] - (P_{b}{}^{\mu}\, \dot{u}_{\mu}) + (\dot{P}_{r}{}^{\mu}\, u_{\mu})\,. \quad \text{(III, 6)}$$

In terms of the four-velocities $\{u^{\lambda}\}$ and $\{\hat{u}^{\lambda}\}$ the Eqs. (III, 5) and (III, 6) read

$$\frac{d}{ds}\left[\frac{(u\,\hat{u})}{2\,\varrho}\right] + \frac{(u\,\hat{\dot{u}})}{6\,\varrho} - \frac{2}{3}\,(\hat{\dot{u}}\,\hat{\dot{u}}) = 0\,, \quad \text{(III, 5a)}$$

$$\frac{d}{ds}\left[\frac{4\,(u\,\hat{u})^2 - 1}{6\,\varrho}\right] - \frac{2}{3}\, \frac{(u\,\hat{u})\,(\hat{u}\,\dot{u})}{\varrho} - \frac{2}{3}\,(\hat{\dot{u}}\,\hat{\dot{u}})\,(u\,\hat{u}) = 0\,. \quad \text{(III, 6a)}$$

From (III, 5a) and (III, 6a) the radiation term is now eliminated by multiplying (III, 5a) with $(u\,\hat{u})$

and subtracting the resulting equation from (III, 6a). After some trivial substitutions one finds

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{(u\,\hat{u})^2-1}{6\,\varrho}\right]-\frac{1}{6}\,\frac{(u\,\hat{u})}{\varrho}\,\frac{\mathrm{d}}{\mathrm{d}s}\,[u\,\hat{u}]+\frac{1}{2}\,\frac{(u\,\hat{u})\,(u\,\hat{\dot{u}})}{\varrho}=0\,. \qquad \text{(III, 7)}$$

Both Eqs. (III, 4) and (III, 7) are frequently used during the following considerations. Since we restrict ourselves on the linear motion, we can put

$$\{u^{\lambda}{}_{(s)}\}=\{\mathrm{Cosh}\,w_{(s)}\,;\ 0,0,\mathrm{Sinh}\,w_{(s)}\} \qquad \text{(III, 8)}$$

and then we will have to prove that the function $w_{(s)}$ must necessarily be a constant throughout the motion.

With the ansatz (III, 8) one easily finds for the scalar products required in the basic Eq. (III, 4) and (III, 7) (abbreviate $\Delta w_{(s)}:=w_{(s)}-w_{(s-\Delta s)}$):

$$(u\,\hat{u})=\mathrm{Cosh}[w_{(s)}-w_{(s-\Delta s)}]\equiv\mathrm{Cosh}\,(\Delta w)\,, \qquad \text{(III, 9a)}$$

$$(\hat{\dot{u}}\,\hat{\dot{u}})=-\hat{\dot{w}}^2;\ \ (\hat{\dot{w}}:=\dot{w}_{(s-\Delta s)})\,, \qquad \text{(III, 9b)}$$

$$(u\,\hat{\dot{u}})=-\hat{\dot{w}}\,\mathrm{Sinh}\,(\Delta w)\,. \qquad \text{(III, 9c)}$$

Substituting these expressions into the basic Eqs. (III, 4, 7), one obtains from (III, 4)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{8\,\mathrm{Cosh}^2\,(\Delta w)+1}{\varrho^2}\right]=-24\,\hat{\dot{w}}^2\,\frac{\mathrm{Cosh}\,(\Delta w)}{\varrho}\,, \qquad \text{(III, 10)}$$

respectively from (III, 7)

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{Sinh}\,(\Delta w)}{\varrho}\right]=3\,\hat{\dot{w}}\,\frac{\mathrm{Cosh}\,(\Delta w)}{\varrho}\,. \qquad \text{(III, 11)}$$

Finally, it is useful to reconsider the invariant electron radius ϱ in the case of linear motion.

Imagine subsidiarily the extension parameter Δs to be variable and differentiate (II, 2c) with respect to Δs

$$\left(\frac{\partial\varrho}{\partial(\Delta s)}\right)_{s=\mathrm{const}}=-\frac{\partial\hat{z}^{\lambda}}{\partial(\Delta s)}\,u_{\lambda}=(\hat{u}^{\lambda}\,u_{\lambda})=\mathrm{Cosh}\,(w-\hat{w})\,.$$

Now, undoing this differentiation by integration leads to

$$\varrho_{(s)}=\int_{\Delta s'=0}^{\Delta s}\mathrm{Cosh}\,[w_{(s)}-w_{(s-\Delta s')}]\,\mathrm{d}(\Delta s')=\int_{s'=s-\Delta s}^{s}\mathrm{Cosh}\,[w_{(s)}-w_{(s')}]\,\mathrm{d}s'\ (\geqq\Delta s)\,. \qquad \text{(III, 12)}$$

IV. The Proof

We proceed in four steps, each of which is characterized by a statement S_j ($j=1,2,3,4$). These statements on the function $w_{(s)}$ [compare (III, 8)], assumed to be continuous together with is first derivative $\mathrm{d}w/\mathrm{d}s$, are proved subsequently and impose increasingly restrictive conditions for $w_{(s)}$, so that finally $w_{(s)}$ turns out to be a constant.

S_1: *The derivative* $\mathrm{d}w/\mathrm{d}s$ *vanishes in the distant future* ($s\to\infty$).

Proof:

Writing equation (III, 4) in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\sqrt{1+8(u\,\hat{u})^2}}{\varrho}\right]=12(\hat{\dot{u}}\,\hat{\dot{u}})\,\frac{(u\,\hat{u})}{\sqrt{1+8(u\,\hat{u})^2}} \qquad \text{(IV, 1)}$$

and integrating this equation from an arbitray time (s) up to the distant future $(s\to\infty)$, one finds

$$\left[\frac{\sqrt{1+8(u\,\hat{u})^2}}{\varrho}\right]_{(s)}=\left[\frac{\sqrt{1+8(u\,\hat{u})^2}}{\varrho}\right]_{(s\to\infty)}-\int_s^{\infty}12\left[(\hat{\dot{u}}\,\hat{\dot{u}})\,\frac{(u\,\hat{u})}{\sqrt{1+8(u\,\hat{u})^2}}\right]_{(s')}\mathrm{d}s'\,. \qquad \text{(IV, 2)}$$

Since the right-hand side of (IV, 1) is negative-semidefinite $[(\dot{u}\,\dot{u})\leqq 0]$, the quantity

$$\frac{\sqrt{1+8(u\,\hat{u})^2}}{\varrho}$$

can only decrease with increasing proper time s; and since this quantity is positive-semidefinite, the difference of this quantity at various times s must be a finite number. Hence, the integral in (IV, 2) converges, which means

$$\lim_{s\to\infty}\left\{(\hat{\dot{u}}\,\hat{\dot{u}})\,\frac{(u\,\hat{u})}{\sqrt{1+8(u\,\hat{u})^2}}\right\}=0\,. \qquad \text{(IV, 3)}$$

Now, because of

$$1\leqq(u\,\hat{u})<\infty$$

we have

$$\frac{1}{3}\leqq\frac{(u\,\hat{u})}{\sqrt{1+8(u\,\hat{u})^2}}<\frac{1}{\sqrt{8}} \qquad \text{(IV, 4)}$$

and therefore the scalar $(\dot{u}\,\dot{u})$ must itself vanish in (IV, 3), i. e. [note (III, 9b)]

$$\lim_{s\to\infty}\dot{w}_{(s)}=0 \qquad \text{(IV, 5)}$$

(q.e.d.).

S_2: *The derivative* $\mathrm{d}w/\mathrm{d}s$ *has an infinite number of zeros with distances* Δs *between two neighbouring members.*

Proof:

First, we conclude from (IV, 5) by the mean value theorem of the calculus of differentiation

$$\Delta w_{(s)} \equiv w_{(s)} - w_{(s-\Delta s)} = \dot{w}_{(\bar{s})} \Delta s\,; \quad s - \Delta s \leqq \bar{s} \leqq s$$

that Δw tends to zero for $s \to \infty$

$$\lim_{s\to\infty} \Delta w_{(s)} = 0\,. \qquad \text{(IV, 6)}$$

This result is used when integrating now Eq. (III, 11) to find

$$\left[\frac{\operatorname{Sinh}(\Delta w)}{\varrho}\right]_{(s)} = -3 \int_s^\infty \hat{w}_{(s')} \frac{\operatorname{Cosh}(\Delta w_{(s')})}{\varrho_{(s')}} \mathrm{d}s'\,. \qquad \text{(IV, 7)}$$

From this equation one recognizes at once, that the derivative $\mathrm{d}w/\mathrm{d}s$ must change sign, at least in one point (say $s_0{}^*$; $s - \Delta s < s_0{}^* < \infty$). For, if it were not so, the right-hand side of (IV, 7) would always assume the opposite sign with respect to the left-hand side (note $\operatorname{Cosh}(\Delta w)/\varrho \geqq 0$). Clearly, the zero in $s_0{}^*$, found above, cannot be the only one, but there must be an infinite number of zeros with change of sign of $\dot{w}_{(s)}$. For, if there would be a finite number of zeros, the latest of which designated with s_{latest}, we could choose $s > s_{\text{latest}}$ in (IV, 7) then repeat the argumentation, which has led to the existence of the first zero in $s_0{}^*$; thus having found a further zero later than the latest one, which is, of course, a contradiction.

Now we can show that any two neighbouring members of this set of zeros have the proper time distance Δs. To this end, the differentiation with respect to s in the basic Eqs. (III, 10) and (III, 11) is carried out explicitly and then the derivative $\mathrm{d}\varrho/\mathrm{d}s$ is eliminated from the two resulting equations. One finds by this procedure

$$\dot{w}_{(s)} = \tfrac{4}{3} \hat{w} \left\{1 + 2 \operatorname{Cosh}^2(\Delta w) + \hat{w}\, \varrho \operatorname{Sinh}(\Delta w)\right\}. \qquad \text{(IV, 8)}$$

This equation tells us, that if $\dot{w}$ vanishes at a certain time (say $s_0{}^*$), then it must vanish at time $s_0{}^* + \Delta s$, too. Continuing this argumentation one gets an infinite set of zeros, spaced with the proper time interval Δs and called $\{s_n{}^*;\ n = 0, 1, 2, \ldots\}$ hereafter (q.e.d.).

S_3: *The function* $w_{(s)}$ *always assumes the same value* (w^*) *in all points* $\{s_n{}^*\}$ *with vanishing derivative* $\mathrm{d}w/\mathrm{d}s$.

Proof:

First we show, that $\mathrm{d}\varrho/\mathrm{d}s$ vanishes in the points $s \in \{s_n{}^*\}$. Writing Eq. (III, 10) in the form

$$8 \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\operatorname{Cosh}^2(\Delta w)}{\varrho^2}\right] + \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\varrho^2}\right] = -24\, \hat{w}^2 \frac{\operatorname{Cosh}(\Delta w)}{\varrho}$$

and Eq. (III, 11) in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\operatorname{Cosh}^2(\Delta w)}{\varrho^2}\right] - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\varrho^2}\right] = 6\, \hat{w} \frac{\operatorname{Cosh}(\Delta w) \operatorname{Sinh}(\Delta w)}{\varrho^2},$$

and then eliminating $\operatorname{Cosh}^2(\Delta w)/\varrho^2$ from these two equations yields

$$\dot{\varrho} = \tfrac{4}{3}\, \varrho\, \hat{w} \operatorname{Cosh}(\Delta w) \left\{\hat{w}\, \varrho + 2 \operatorname{Sinh}(\Delta w)\right\}. \qquad \text{(IV, 9)}$$

From this equation, together with the fact that $\dot{w}$ vanishes for $s \in \{s_n{}^*\}$, it follows that $\dot{\varrho}$ vanishes as well in the points $\{s_n{}^*;\ n = 1, 2, 3, \ldots\}$:

$$\dot{\varrho}_{(s_n{}^*)} = 0\,. \qquad \text{(IV, 10)}$$

Now, calculate $\dot{\varrho}$ by means of (III, 12) and find

$$\dot{\varrho}_{(s)} = \dot{w}_{(s)} \int_{s'=s-\Delta s}^{s} \operatorname{Sinh}\left[w_{(s)} - w_{(s')}\right] \mathrm{d}s' + 1 - \operatorname{Cosh}(\Delta w)\,. \qquad \text{(IV, 11)}$$

We therefore have in the points $\{s_n{}^*\}$

$$\operatorname{Cosh}(\Delta w) = 1\,,$$

which means

$$w_{(s_n{}^*)} = w_{(s^*{}_{n-1})} \equiv w^* \quad (n = 1, 2, 3, \ldots) \qquad \text{(IV, 12)}$$

(q.e.d.).

S_4: *The function* $w_{(s)}$ *is constant.*

Proof:

It is most convenient to begin this last and essential step with the proof of the auxiliary statement, that there is at least one further zero between two arbitrary zeros of $\mathrm{d}w/\mathrm{d}s$. To show this, we specialize the time s in (IV, 7) to be one point of the set $\{s_n{}^*;\ n = 1, 2, 3, \ldots\}$. Because of (IV, 12) we then know, that

$$\Delta w_{(s_n{}^*)} \equiv w_{(s_n{}^*)} - w_{(s^*{}_{n-1})} = 0 \qquad \text{(IV, 12a)}$$

is valid and therefore the left-hand side of (IV, 7) must vanish:

$$0 = -3 \int_{s'=s_n^*}^{\infty} \hat{\dot{w}}_{(s')} \frac{\operatorname{Cosh}(\Delta w_{(s')})}{\varrho_{(s')}} \, ds' . \quad \text{(IV, 13)}$$

Taking here two arbitrary neighbouring n, one gets

$$0 = \int_{s'=s_n^*}^{s^*_{n+1}} \hat{\dot{w}}_{(s')} \frac{\operatorname{Cosh}(\Delta w_{(s')})}{\varrho_{(s')}} \, ds' . \quad \text{(IV, 14)}$$

Because of the positive-semidefiniteness of the expression $\operatorname{Cosh}(\Delta w)/\varrho$ one concludes from (IV, 14), that $\dot{w}$ must have a further zero in the interior of the interval $\langle s^*_{n-1} \mid s_n^* \rangle$, i. e. for $s^*_{n-1} < s < s_n^*$. Let us call this intermediate zero s^{**}_{n-1}. The same reasoning as below (IV, 8) now leads from one new zero s_n^{**} to a whole set $\{s_n^{**}; n = 0, 1, 2, \ldots\}$ of infinitely many zeros spaced equidistantly with Δs on the proper time axis, where $s_n^* < s_n^{**} < s^*_{n+1}$ for each $n = 0, 1, 2, 3, \ldots$ It is essential that the intersection of the two sets $\{s_n^*\}$ and $\{s_n^{**}\}$ is empty. In full analogy to the considerations below S_3 [especially Eq. (IV, 9) and (IV, 11)] we can argue that the function $w_{(s)}$ always assumes the same value (say w^{**}) in the points of the set $\{s_n^{**}\}$:

$$\Delta w_{(s_n^{**})} = 0 . \quad \text{(IV, 15)}$$

Now we can construct a third set of zeros $\{s_n^{3*}\}$ with $s_n^* < s_n^{3*} < s_n^{**}$ $(n = 0, 1, 2, 3, \ldots)$ by integrating the basic Eq. (III, 11) between two neighbouring zeros s_n^* and s_n^{**} belonging to the two different sets mentioned above. This yields

$$\left[\frac{\operatorname{Sinh}(\Delta w)}{\varrho}\right]_{(s_n^{**})} - \left[\frac{\operatorname{Sinh}(\Delta w)}{\varrho}\right]_{(s_n^*)} = \int_{s'=s_n^*}^{s_n^{**}} 3\, \hat{\dot{w}}_{(s')} \frac{\operatorname{Cosh}(\Delta w_{(s')})}{\varrho_{(s')}} \, ds' . \quad \text{(IV, 16)}$$

But with (IV, 12a) and (IV, 15) the left-hand side of (IV, 16) vanishes

$$0 = \int_{s'=s_n^*}^{s_n^{**}} \hat{\dot{w}}_{(s')} \frac{\operatorname{Cosh}(\Delta w_{(s')})}{\varrho_{(s')}} \, ds' . \quad \text{(IV, 17)}$$

Equation (IV, 17) is quite analogous to (IV, 14) and so are the conclusions drawn from it: There exists an infinite set of zeros $\{s_n^{(3*)}\}$, as indicated above, with deplete intersection with respect to the first two sets. Obviously, one can continue this procedure and would then find an infinite number of infinite sets $\{s_n^{(m*)}\}$ of zeros of dw/ds, whereby all these sets have properties equal to those of $\{s_n^*\}$ or $\{s_n^{**}\}$ discussed above.

Summarizing results, we can formulate the following lemma:

Between any two zeros of dw/ds *there is a further zero, which does not coincide with the two original zeros.*

Clearly, we can take advantage of this fact in order to complete our proof.

We choose an arbitrary time s and shall show, that in this arbitrary point s the derivative dw/ds vanishes. Surely, the point s is situated between two members of suitable sets of zeros, say

$$s_n^{(m*)} < s < s_l^{(j*)}$$

with n, m, l, j being natural numbers. We can shorten the length $|s_n^{(m*)} - s_l^{(j*)}|$ of the interval $\langle s_n^{(m*)} \mid s_l^{(j*)} \rangle$ by applying the above lemma to construct a zero $s^{(1)}$ with $s_n^{(m*)} < s^{(1)} < s_l^{(j*)}$. $s^{(1)}$ divides the original interval $\langle s_n^{(m*)} \mid s_l^{(j*)} \rangle$ into two parts. Consider only that part, in which s is situated. Without loss of generality we may put $s^{(1)} < s < s_l^{(j*)}$ with $|s^{(1)} - s_l^{(j*)}| < |s_n^{(m*)} - s_l^{(j*)}|$.

This process of shortening the intervals, in which s is situated, can be continued and there is obviously no lower bound for the lengths of the shortened intervals. Therefore the intervals shrink on to the point s (Cantor's axiom) and since the interval end points have $\dot{w} = 0$, this must be true for the arbitrarily chosen point s. Hence $dw/ds = 0$ for all times and $w_{(s)}$ is constant (q.e.d.).

So far, we have only shown the vanishing of $\dot{w}$ for $s \geqq s_0^*$, but we can repeat, under this condition, all the preceding argumentation if $\dot{w} \neq 0$ is assumed for $s < s_0^*$ and would then find that also for times earlier than s_0^* (dw/ds) must vanish.

V. Discussion

As can be seen from the preceding section, the runaway solutions are not present in the finite-size theory of the electron but they come in artificially in the point particle limit (II, 4). Therefore we must conclude that the totality of solutions of the finite-size theory is not approximated by the totality of solutions to the Lorentz-Dirac Equation (II, 4). The latter set is the larger one, the difference with respect to the first set consisting of all possible runaway solutions, which can be superimposed on the physically reasonable solutions of the Lorentz-Dirac equation.

Therefore, one might think that the solutions of the integro-differential form of the Lorentz-Dirac equation

$$m_0 c^2 \dot{u}^{\lambda}{}_{(s)} = \int_{s'=s}^{\infty} \exp\{(s-s')/s_0\} \cdot \{Z F_e{}^{\mu\lambda} u_\mu + \tfrac{2}{3} Z^2 (\dot{u}\,\dot{u}) u^\lambda\}_{(s')} \frac{ds'}{s_0} \quad (V,1)$$

in which there are no runaway solutions, are the point-like approximation of the finite-size solutions.

Indeed, Eq. (V, 1) was proposed by Rohrlich [1] to be the proper equation of motion instead of (II, 4) in order to ensure the asymptotic conditions of vanishing acceleration for the initial and final states. In the present context, we prefer (V, 1) to (II, 4) for the sake of comparison with the finite-size theory: the real electron is assumed to be of finite-size, which prohibits runaway solutions, and the point-limit solutions are not allowed to bring in completely new features missing in the more general theory of finite-size.

There is a concrete type of motion supporting this point of view: this is the hyperbolic case. We have studied this case [3] for the finite-size electron and have found the phenomenon of a critical acceleration being of the order $\dot{w} \approx r_0^{-1}$. This critical value cannot be surpassed. Now, the same phenomenon occurs in the point limit, if (V, 1) is assumed to be the equation of motion rather than (II, 4) (see Appendix).

Appendix

Hyperbolic Motion in the Lorentz-Dirac Theory

A special type of motion has ever been a sort of puzzle in the Lorentz-Dirac theory (II, 4) of the radiating point electron: this is the hyperbolic case, whereby the particle moves in a constant, homogeneous electric field E $(=F^{03} = -F^{30})$ with no magnetic field present parallel to the field vector $\mathbf{E}$.

To deal with this case, people [1] usually insert the linear ansatz (III, 8) into the equation of motion (II, 4) and obtain

$$m_0 c^2 \dot{w} = Z E + \tfrac{2}{3} Z^2 \ddot{w} \quad (A,1)$$

or

$$\dot{w} - s_0 \ddot{w} = Z E / m_0 c^2 \quad (A,1a)$$

with

$$s_0 = \tfrac{2}{3} Z^2 / m_0 c^2 = \tfrac{4}{3} r_0 ,$$

where r_0 is the classical electron radius. In order to exclude runaway solutions, which satisfy the homogeneous equation, one converts (A, 1) into an integro-differential equation

$$m_0 c^2 \dot{w}_{(s)} = Z \int_{s'=s}^{\infty} E_{(s')} \exp\{(s-s')/s_0\} \frac{ds'}{s_0}. \quad (A,1b)$$

This equation is then considered as the primary equation of motion in the one-dimensional case.

For the hyperbolic type of motion with $E = \text{const}$, one finds readily from (A, 1b)

$$m_0 c^2 \dot{w} = Z E = \text{const}, \quad (A,2)$$

which is the same as for a neutral particle described by

$$m_0 c^2 \dot{u}^\lambda = K^\lambda \quad (A,3)$$

with

$$K^\lambda = \left\{ \frac{\mathbf{K}\cdot\mathbf{V}/c}{\sqrt{1-\mathbf{V}^2/c^2}} ; \frac{\mathbf{K}}{\sqrt{1-\mathbf{V}^2/c^2}} \right\}$$

and

$$|\mathbf{K}| = Z E .$$

This curious fact is rather astonishing: Why does the radiation recoil not retard the radiating particle [decribed by (A, 1)] with respect to the non-radiating particle [described by (A, 3)]?

Moreover, we are interested here in the question, whether the Lorentz-Dirac theory, expressed by (A, 1) or by Rohrlich's equation (A, 1b), does exhibit some critical value for the invariant acceleration k $(\equiv \dot{w} = \text{const})$ as is the case for the finite-size electron?

Obviously, there is no critical effect in Rohrlich's treatment (A, 1b). Therefore let us attack these two questions in a new way!

According to our point of view in Section V, the physically reasonable solutions in the point-like theory are given by the solutions of the integro-differential Equation (V, 1). It should therefore not be allowed to insert the ansatz (III, 8) into (II, 4) and to convert the resulting Eq. (A, 1) in an integro-differential Eq. (A, 1b); rather (III, 8) has to be inserted into the primary Eq. (V, 1) which yields

$$m_0 c^2 \dot{w}_{(s)} v^\lambda{}_{(s)} = \int_s^\infty \exp\{(s-s')/s_0\} \cdot \{Z E_{(s')} v^\lambda{}_{(s')} - \tfrac{2}{3} Z^2 \dot{w}^2{}_{(s')} u^\lambda{}_{(s')}\} \frac{ds'}{s_0}, \quad (A,4)$$

where we have abbreviated the principal normal to the world line by $v^\lambda{}_{(s)}$:

$$\dot{u}^\lambda{}_{(s)} = \dot{w}_{(s)} v^\lambda{}_{(s)} = \dot{w}_{(s)} \{\text{Sinh}\, w_{(s)};\ 0, 0, \text{Cosh}\, w_{(s)}\}. \quad (A,5)$$

One can split up now the vector Eq. (A, 4) into two scalar equations by multiplying with $v_{\lambda(s)}$ respectively $u_{\lambda(s)}$. In doing so, the following relations are useful to introduce $(w' := w_{(s')})$:

$$v^{\lambda}{}_{(s')}\, v_{\lambda(s)} = -\operatorname{Cosh}(w'-w)\,, \qquad (A, 6a)$$

$$u^{\lambda}{}_{(s')}\, v_{\lambda(s)} = -\operatorname{Sinh}(w'-w)\,, \qquad (A, 6b)$$

$$u^{\lambda}{}_{(s')}\, u_{\lambda(s)} = \operatorname{Cosh}(w'-w)\,. \qquad (A, 6c)$$

Now one gets the two scalar equations

$$m_0 c^2 \dot{w}_{(s)} = \int_s^\infty \exp\{(s-s')/s_0\}\{Z E_{(s')} \operatorname{Cosh}(w'-w) - \tfrac{2}{3} Z^2 \dot{w}^2{}_{(s')} \operatorname{Sinh}(w'-w)\} \frac{\mathrm{d}s'}{s_0}, \qquad (A, 7a)$$

$$0 = \int_s^\infty \exp\{(s-s')/s_0\}\{Z E_{(s')} \operatorname{Sinh}(w'-w) - \tfrac{2}{3} Z^2 \dot{w}^2{}_{(s')} \operatorname{Cosh}(w'-w)\} \frac{\mathrm{d}s'}{s_0}. \qquad (A, 7b)$$

Clearly, one verifies easily by differentiation that each solution $w_{(s)}$ of (A, 7) satisfies (A, 1) but the inverse requires a moment's thought.

Consider the hyperbolic motion type (A, 2) and observe the following integrals needed in (A, 7) $(k \equiv \dot{w} = \text{const})$

$$\int_{s'=s}^\infty \exp\{(s-s')/s_0\} \cdot \operatorname{Sinh}[k(s'-s)] \frac{\mathrm{d}s'}{s_0} = \frac{k s_0}{1-k^2 s_0^2} \qquad (A, 8a)$$

$$\int_{s'=s}^\infty \exp\{(s-s')/s_0\} \cdot \operatorname{Cosh}[k(s'-s)] \frac{\mathrm{d}s'}{s_0} = \frac{1}{1-k^2 s_0^2}\,. \qquad (A, 8b)$$

Both integrals (A, 8) do only exist, if $|k| < 1/s_0$. Hence hyperbolic motion is only a solution to the equation of motion (V, 1) if this condition is satisfied; i. e. there is a critical invariant acceleration k_c

$$|k_c| = 1/s_0 = \tfrac{3}{4} r_0^{-1}\,. \qquad (A, 9a)$$

This effect was obtained by adopting (V, 1) as the equation of motion instead of (A, 1).

The result thus obtained should be compared with the critical acceleration of the finite-size electron [3], being found as

$$|k_c| \approx 1.4\, r_0^{-1}\,. \qquad (A, 9b)$$

Apart from a numerical factor of the magnitude 1, both results (A, 9) exhibit the same phenomenon: the existence of a critical acceleration of the radiating electron in the region $k \approx r_0^{-1}$.

The singular behaviour in the case $k \to k_c$ is further elucidated by a study of the radiation recoil: With (A, 8) the equation (A, 7b) confirms (A, 2) but (A, 7a) yields

$$m_0 c^2 \dot{w}_{(s)} = Z E \frac{1}{1-k^2 s_0^2} - \frac{2}{3} Z^2 k^2 \frac{k s_0}{1-k^2 s_0^2}\,. \qquad (A, 10)$$

From here it is seen that the Schott-term, which has been identified [2, 3] as the energy-momentum of the distortion of the Coulomb field by virtue of retardation in the case of an accelerated motion, has led to an effective force emerging on the right-hand side of (A, 10). But we know that the real external force is ZE [as shown in (A, 1)], so we cast the common factor $(1-k^2 s_0^2)^{-1}$ from the right in (A,10) to the left

$$m_0 c^2 (1-k^2 s_0^2)\, \dot{w}_{(s)} = Z E - \tfrac{2}{3} Z^2 k^3 s_0 \qquad (A, 11)$$

and observe now the emergence of an effective mass

$$m_{\text{eff}} = m_0 (1-k^2 s_0^2) \qquad (A, 12)$$

into which the Schott energy has been absorbed. From (A, 11) we clearly recognize the mechanism of hyperbolic motion. The usual Lorentz force (ZE) together with the radiation recoil $(\tfrac{2}{3} Z^2 k^3 s_0)$ produce the same acceleration k as is exhibited by the non-radiating reference particle (A, 3) in the same force field, because the inertia of the radiating particle is diminished under the influence of the Schott energy. This expresses Einstein's idea that every form of energy (here Schott energy) of a body contributes to its inertia. (A, 12) exhibits again the singular behaviour of the radiating particle in the case $k \to k_c$: the effective mass vanishes for the critical value k_c!

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